

SYMPLECTIC AND HYPERKÄHLER IMPLOSION

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ABSTRACT. We review the quiver descriptions of symplectic and hyperkähler implosion in the case of $SU(n)$ actions. We give quiver descriptions of symplectic implosion for other classical groups, and discuss some of the issues involved in obtaining a similar description for hyperkähler implosion.

1. INTRODUCTION

Symplectic implosion is an abelianisation construction in symplectic geometry invented by Guillemin, Jeffrey and Sjamaar [11]. Given a symplectic manifold M with a Hamiltonian action of a compact group K , its imploded cross-section M_{impl} is a symplectic stratified space with a Hamiltonian action of a maximal torus T of K , such that the symplectic reductions of M by K agree with the symplectic reductions of the implosion by T .

There is a universal example of symplectic implosion, obtained by taking M to be the cotangent bundle T^*K . The imploded space $(T^*K)_{\text{impl}}$ carries a Hamiltonian torus action for which the symplectic reductions are the coadjoint orbits of K . It also carries a Hamiltonian action of K which commutes with the T action, and the implosion M_{impl} of any symplectic manifold M with a Hamiltonian action of K can be constructed as the symplectic reduction at 0 of the product $M \times (T^*K)_{\text{impl}}$ by the diagonal action of K .

The universal symplectic implosion $(T^*K)_{\text{impl}}$ can also be described in a more algebraic way, as the geometric invariant theory quotient $K_{\mathbb{C}}//N$ of the complexification $K_{\mathbb{C}}$ of K by a maximal unipotent subgroup N . This is the affine variety $\text{Spec}(\mathcal{O}(K_{\mathbb{C}})^N)$ associated to the algebra of N -invariant regular functions on $K_{\mathbb{C}}$, and may also be described as the canonical affine completion of the orbit space $K_{\mathbb{C}}/N$ which is a dense open subset of $K_{\mathbb{C}}//N$.

Many constructions in symplectic geometry involving the geometry of moment maps have analogues in hyperkähler geometry. We recall here that a hyperkähler structure is given by a Riemannian metric g and a triple of complex structures satisfying the quaternionic relations. In fact we then acquire a whole two-sphere's worth of complex structures, parametrised by the unit sphere in the imaginary quaternions. The

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metric is required to be Kähler with respect to each of the complex structures. In this way a hyperkähler structure defines a two-sphere of symplectic structures.

Just as the cotangent bundle T^*K of a compact Lie group carries a natural symplectic structure, so, by work of Kronheimer, the cotangent bundle $T^*K_{\mathbb{C}}$ of the complexified group carries a hyperkähler structure [19]. Moreover, in a series of papers Kronheimer, Biquard and Kovalev showed that the coadjoint orbits of $K_{\mathbb{C}}$ admit hyperkähler structures [20, 21, 2, 16]. These orbits are not however closed in $\mathfrak{k}_{\mathbb{C}}^*$ (and the hyperkähler metrics are not complete) except in the case of semisimple orbits.

In [5] and subsequent papers [6, 7] we developed a notion of a universal hyperkähler implosion $(T^*K_{\mathbb{C}})_{\text{hkipl}}$ for $SU(n)$ actions. The hyperkähler implosion of a general hyperkähler manifold M with a Hamiltonian action of $K = SU(n)$ can then be defined as the hyperkähler quotient of $M \times (T^*K_{\mathbb{C}})_{\text{hkipl}}$ by the diagonal action of K . As in the symplectic case the universal hyperkähler implosion carries an action of $K \times T$ where $K = SU(n)$ and T is its standard maximal torus. As coadjoint orbits for the complex group are no longer closed in general, and are not uniquely determined by eigenvalues, the hyperkähler quotients of $(T^*K_{\mathbb{C}})_{\text{hkipl}}$ by the torus action need not be single orbits. Instead, they are the Kostant varieties, that is, the varieties in $\mathfrak{sl}(n, \mathbb{C})^*$ obtained by fixing the values of the invariant polynomials for this Lie algebra. These varieties are unions of coadjoint orbits and are closures in $\mathfrak{sl}(n, \mathbb{C})^*$ of the regular coadjoint orbits of $K_{\mathbb{C}} = SL(n, \mathbb{C})$. We refer to [4], [15] for more background on the Kostant varieties.

Again by analogy with the symplectic case, we can describe the hyperkähler implosion in terms of geometric invariant theory (GIT) quotients by nonreductive group actions. Explicitly, the implosion is $(SL(n, \mathbb{C}) \times \mathfrak{n}^0) // N$ where N is a maximal unipotent subgroup of $K_{\mathbb{C}} = SL(n, \mathbb{C})$ and \mathfrak{n}^0 is the annihilator in $\mathfrak{sl}(n, \mathbb{C})^*$ of its Lie algebra \mathfrak{n} . Thus the universal hyperkähler implosion for $K = SU(n)$ can be identified with the complex-symplectic quotient $(SL(n, \mathbb{C}) \times \mathfrak{n}^0) // N$ of $T^*SL(n, \mathbb{C})$ by N in the GIT sense, just as the symplectic implosion is the GIT quotient of $K_{\mathbb{C}}$ by N .

In the case of $K = SU(n)$ dealt with in [5], it is possible to describe the hyperkähler implosion via a purely finite-dimensional construction using quiver diagrams. This construction was motivated by a quiver description of the symplectic implosion for $SU(n)$ we described in §4 of [5].

In this article we shall extend these results concerning symplectic implosion to other classical groups, that is the special orthogonal and symplectic groups. Our approach will be inspired by the description by Lian and Yau [23] of coadjoint orbits for compact classical groups using quivers. This suggests ways to extend the quiver construction of

the universal hyperkähler implosion from the case of $SU(n)$ to general classical groups, and we discuss some of the issues involved here.

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2. SYMPLECTIC QUIVERS

We begin by trying to construct a quiver model for the universal symplectic implosion in the case of the orthogonal and symplectic groups, as was done in [5] for special unitary groups.

We consider diagrams of vector spaces and linear maps

$$(2.1) \quad 0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{r-2}} V_{r-1} \xrightarrow{\alpha_{r-1}} V_r = \mathbb{C}^n.$$

The dimension vector is defined to be $\mathbf{n} = (n_1, \dots, n_{r-1}, n_r = n)$ where $n_i = \dim V_i$. We will say that the representation is *ordered* if $0 \leq n_1 \leq n_2 \leq \cdots \leq n_r = n$ and *strictly ordered* if $0 < n_1 < n_2 < \cdots < n_r = n$.

In [5] we considered $V_r = \mathbb{C}^n$ as a representation of $SU(n)$, or its complexification $SL(n, \mathbb{C})$. In this setting we say the quiver is *full flag* if $r = n$ and $n_i = i$ for each i . We took the geometric invariant theory quotient of the space of full flag quivers by $SL := \prod_{i=1}^{r-1} SL(V_i)$ (or equivalently, the symplectic quotient by $\prod_{i=1}^{r-1} SU(V_i)$). The stability conditions imply that the quiver decomposes into a quiver with zero maps and a quiver with all maps injective. It was therefore sufficient to analyse the injective quivers up to equivalence. We found that the quotient could be stratified into 2^{n-1} strata, indexing the flag of dimensions of the injective quivers (after the quivers had been contracted to remove edges where the maps were isomorphisms). Equivalently, the strata were indexed by the ordered partitions of n . Each stratum could be identified with $SL(n, \mathbb{C})/[P, P]$ where P is the parabolic associated to the given flag. The upshot was that the full GIT quotient can be identified as an affine variety with the affine completion $SL(n, \mathbb{C})//N = \text{Spec } \mathcal{O}(SL(n, \mathbb{C}))^N$ of the open stratum $SL(n, \mathbb{C})/N$.

We now wish to view V_r as a representation of an orthogonal or symplectic group. This involves introducing the associated bilinear forms. Our approach will be motivated by the description due to Lian and Yau in [23] of a quiver approach to generalised flag varieties for symplectic and orthogonal groups.

Note that for consistency with [5] we have altered the notation of [23] in some respects. In particular we use r rather than $r + 1$ for the top index, and we use n_i rather than d_i for the dimensions.

In the orthogonal case, we let J denote the matrix with entries

$$J_{ij} = \delta_{n+1-i,j} \quad (1 \leq i, j \leq n)$$

which are 1 on the antidiagonal and 0 elsewhere. We therefore have on \mathbb{C}^n a symmetric bilinear form $B(v, w) = v^t J w$ where v^t denotes the transpose of v , which is preserved by

$$SO(n, \mathbb{C}) = \{ g : g^t J g = J \}$$

We note that the condition for h to be in the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ is $h^t J + J h = 0$, that is, that h is skew-symmetric about the ANTI-diagonal. In particular, h may have arbitrary elements in the top left $d \times d$ block as long as $d \leq \frac{n}{2}$.

Motivated by [23] let us now consider ordered diagrams where

$$n_{r-1} \leq \frac{n}{2}$$

and we impose on α_{r-1} the condition

$$(2.2) \quad \alpha_{r-1}^t J \alpha_{r-1} = 0.$$

Equivalently, this is the condition that the image of α_{r-1} be an isotropic subspace of \mathbb{C}^n with respect to J (which is the reason for the inequality above). The space of α_{r-1} satisfying this condition is of dimension $nn_{r-1} - \frac{1}{2}n_{r-1}(n_{r-1} + 1)$.

We let $R(\mathbf{n})$ be the space of all such diagrams satisfying (2.2) with dimension vector \mathbf{n} .

Observe that the complexification $GL := \prod_{i=1}^{r-1} GL(V_i)$ of $\tilde{H} := \prod_{i=1}^{r-1} U(V_i)$ acts on $R(\mathbf{n})$ by

$$\begin{aligned} \alpha_i &\mapsto g_{i+1} \alpha_i g_i^{-1} \quad (i = 1, \dots, r-2), \\ \alpha_{r-1} &\mapsto \alpha_{r-1} g_{r-1}^{-1}. \end{aligned}$$

There is also a commuting action of $SO(n, \mathbb{C})$ by left multiplication of α_{r-1} ; note that the full group $GL(n, \mathbb{C})$ does not now act because it does not preserve (2.2).

As in [5], we shall study the symplectic quotient of $R(\mathbf{n})$ by the action of

$$H := \prod_{i=1}^{r-1} SU(V_i)$$

or equivalently the GIT quotient of $R(\mathbf{n})$ by its complexification

$$H_{\mathbb{C}} = SL := \prod_{i=1}^{r-1} SL(V_i),$$

viewed as a subgroup of GL in the obvious way. This quotient will have residual actions of the $r-1$ -dimensional compact torus $T^{r-1} = (S^1)^{r-1}$ and its complexification, as well as of $SO(n, \mathbb{C})$.

Let us observe that the dimension $nn_{r-1} - \frac{1}{2}n_{r-1}(n_{r-1} + 1)$ of the set of α_{r-1} satisfying (2.2) equals the dimension of the coset space $SO(n, \mathbb{C}) / SO(n - n_{r-1}, \mathbb{C})$. In fact, in the orthogonal case with n odd, we can show that this coset space equals the set of injective α_{r-1} satisfying (2.2).

For if n is odd $SO(n, \mathbb{C})$ acts transitively on the set of isotropic subspaces of \mathbb{C}^n of fixed dimension, so α_{r-1} can be put into the form

$$\begin{pmatrix} A_{n_{r-1} \times n_{r-1}} \\ 0_{(n-n_{r-1}) \times n_{r-1}} \end{pmatrix}$$

via the $SO(n, \mathbb{C})$ action. As $n_{r-1} \leq \frac{n}{2}$, we can consider matrices in $SO(n, \mathbb{C})$ with an arbitrary invertible $n_{r-1} \times n_{r-1}$ block in the top left corner and a zero $(n - n_{r-1}) \times n_{r-1}$ block in the lower left. So in fact α_{r-1} can be put into the standard form (used also in the A_n case in [5])

$$\begin{pmatrix} I_{n_{r-1} \times n_{r-1}} \\ 0_{(n-n_{r-1}) \times n_{r-1}} \end{pmatrix}.$$

The connected component of the stabiliser of this configuration for the $SO(n, \mathbb{C})$ action is $SO(n - n_{r-1}, \mathbb{C})$.

We now obtain a description of quivers in $R(\mathbf{n})$ with all α_i injective, modulo the action of SL . For, combining the above observation with the arguments of §4 of [5], the action of $SL \times SO(n, \mathbb{C})$ can be used to put the maps in standard form

$$\alpha_i = \begin{pmatrix} I_{n_i \times n_i} \\ 0_{(n_{i+1}-n_i) \times n_i} \end{pmatrix}.$$

The remaining freedom is a commutator of a parabolic in $SO(n, \mathbb{C})$ where the first $r - 1$ block sizes are $n_{i+1} - n_i$. The blocks in the Levi subgroup lie in $SL(n_{i+1} - n_i)$ which is why we get the commutator rather than the full parabolic. Hence the injective quivers with fixed dimension vector \mathbf{n} modulo the action of SL are parametrised by $SO(n, \mathbb{C})/[P, P]$, where P is the parabolic associated to the dimension vector.

Note that the blocks corresponding to the upper left square of size $n_{r-1} = \sum_{i=0}^{r-2} n_{i+1} - n_i$ will determine the blocks in the lower right square of size n_{r-1} , at least on Lie algebra level, by the property of being in the orthogonal group.

Remark 2.3. By intersecting the parabolic (rather than its commutator) with the compact group $SO(n)$ we get $\prod_{i=0}^{r-2} U(n_{i+1} - n_i) \times SO(n - n_{r-1})$, which is the isotropy group for the associated compact flag variety. Putting $p_{i+1} = n_{i+1} - n_i$, and $\ell = n - n_{r-1}$, we get $\sum_{i=1}^{r-1} p_i = n - \ell$, in accordance with the results of [1, p. 233, section 8H].

For a model for the non-reductive GIT quotient $K_{\mathbb{C}}//N$ in the B_k case, that is when $n = 2k + 1$ and $K = SO(2k + 1)$, we can take \mathbf{n} equal to $(1, 2, 3, \dots, k, 2k + 1)$ which will be the full flag condition in this context. We now consider the GIT quotient $R(\mathbf{n})//SL$. Note that at this stage the maps α_i are not assumed to be injective.

As the SL action is the same as in the A_n case, the stability analysis proceeds as in [5]. We find that for polystable configurations we may decompose each vector space \mathbb{C}^i as

$$(2.4) \quad \mathbb{C}^i = \ker \alpha_i \oplus \mathbb{C}^{m_i},$$

where $\mathbb{C}^{m_i} = \text{im } \alpha_{i-1}$ if $m_i \neq 0$. So the quiver decomposes into a zero quiver and an injective quiver. After contracting legs of the quiver which are isomorphisms, as in [5], we obtain a strictly ordered injective quiver of the form considered above. We stratify the quotient $R(\mathbf{n})//SL$ by the flag of dimensions of the injective quiver, as in the $SL(n, \mathbb{C})$ case.

We have thus identified the strata of the GIT quotient of the space of full flag quivers with the strata $SO(n, \mathbb{C})/[P, P]$ of the universal symplectic implosion, or equivalently the non-reductive GIT quotient $K_{\mathbb{C}}//N$ (where N is a maximal unipotent subgroup). As the complement of the open stratum $SO(n, \mathbb{C})/N$ is of complex codimension strictly greater than one, we see that the implosion $K_{\mathbb{C}}//N$ and the GIT quotient of the space of full flag quivers are affine varieties with the same coordinate ring $\mathcal{O}(SO(n, \mathbb{C}))^N$, and so are isomorphic.

Guillemin, Jeffrey and Sjamaar [11] showed that the non-reductive GIT quotient $K_{\mathbb{C}}//N$ has a $K \times T$ -invariant Kähler structure such that it can be identified symplectically with the universal symplectic implosion for K . In order to see this Kähler structure on $R(\mathbf{n})//SL$ we can put an $\tilde{H} \times K$ -invariant flat Kähler structure on $R(\mathbf{n})$ and identify the GIT quotient $R(\mathbf{n})//SL$ with the symplectic quotient $R(\mathbf{n})//H$. To achieve $\tilde{H} \times K$ -invariance we use the standard flat Kähler structure on $(\mathbb{C}^{j-1})^* \otimes \mathbb{C}^j$ for $j \leq r-1$ but the flat Kähler structure defined by J on

$$(\mathbb{C}^{r-1})^* \otimes \mathbb{C}^n \cong (\mathbb{C}^n)^{r-1}.$$

Recall that a polystable quiver decomposes into the sum of a zero quiver and an injective quiver, and determines for us a partial flag in \mathbb{C}^n

$$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{r-1} \subseteq \mathbb{C}^n,$$

where $W_j = \text{im } \alpha_{r-1} \circ \alpha_{r-2} \circ \cdots \circ \alpha_j$, whose dimension vector ($w_1 = \dim W_1, \dots, w_{r-1} = \dim W_{r-1}$) is determined by the injective summand. The condition that $\alpha_{r-1}^t J \alpha_{r-1} = 0$ ensures that W_{r-1} is an isotropic subspace of \mathbb{C}^n , so we can use the action of the compact group $K = SO(n)$ to put this flag into standard form with W_j spanned by the first w_j vectors in the standard basis for \mathbb{C}^n . Then we can use the action of

SL to put the polystable quiver into the form where

$$(2.5) \quad \alpha_j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \nu_1^j & 0 & \dots & 0 \\ 0 & \nu_2^j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_j^j \end{pmatrix}$$

for $j < r - 1$ and

$$\alpha_{r-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \nu_1^{r-1} & 0 & \dots & 0 \\ 0 & \nu_2^{r-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_{r-1}^{r-1} \end{pmatrix}$$

with $\nu_i^j \in \mathbb{C}$. Let $R_T(\mathbf{n})$ denote the subspace of $R(\mathbf{n})$ consisting of quivers of this form.

Note that the moment map for the action of the unitary group $U(V_j)$ on $R(\mathbf{n})$ takes a quiver (2.1) to

$$\bar{\alpha}_j^t \alpha_j - \alpha_{j-1} \bar{\alpha}_{j-1}^t$$

for $1 \leq j \leq r - 1$, so the moment map for the action of the product $\tilde{H} = \prod_{j=1}^{r-1} U(V_j)$ takes $R_T(\mathbf{n})$ into the Lie algebra of the product of the standard (diagonal) maximal tori T_{V_j} of the unitary groups $U(V_j)$. Thus there is a natural map of symplectic quotients

$$\theta_T: R_T(\mathbf{n}) // H_T \rightarrow R(\mathbf{n}) // H$$

where $H_T = \prod_{j=1}^{r-1} (T_{V_j} \cap SL(V_j))$ is a maximal torus of H . Moreover

$$R(\mathbf{n}) // H = K \theta_T(R_T(\mathbf{n}) // H_T)$$

where $K = SO(n)$ and $R_T(\mathbf{n}) // H_T$ is a toric variety.

The moment map for the action of the torus $T^{r-1} = (S^1)^{r-1}$ on $R(\mathbf{n}) // H$ takes a point represented by a quiver of the form (2.5) satisfying the moment map equations

$$\begin{pmatrix} |\nu_1^j|^2 & 0 & \dots & 0 \\ 0 & |\nu_2^j|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\nu_j^j|^2 \end{pmatrix} = \lambda_j^{\mathbb{R}} I + \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & |\nu_1^{j-1}|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\nu_{j-1}^{j-1}|^2 \end{pmatrix}$$

for some $\lambda_1^{\mathbb{R}}, \dots, \lambda_{r-1}^{\mathbb{R}} \in \mathbb{R}$, or equivalently

$$(2.6) \quad |\nu_i^j|^2 = \lambda_j^{\mathbb{R}} + \lambda_{j-1}^{\mathbb{R}} + \dots + \lambda_{j-i+1}^{\mathbb{R}} \quad \text{if } 1 \leq i \leq j < n,$$

to $(\lambda_1^{\mathbb{R}}, \dots, \lambda_{r-1}^{\mathbb{R}})$ in the Lie algebra of T^{r-1} , while the moment map for the action of K takes this point to

$$\begin{pmatrix} -|\nu_{r-1}^{r-1}|^2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & -|\nu_1^{r-1}|^2 & 0 & \dots & 0 \\ 0 & \dots & 0 & |\nu_1^{r-1}|^2 & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & |\nu_{r-1}^{r-1}|^2 \end{pmatrix},$$

up to constant scalar factors depending on conventions. The image of the toric variety $R_T(\mathbf{n})//H_T$ under this moment map is the positive Weyl chamber \mathfrak{t}_+ of $K = SO(n)$, and we obtain a symplectic identification of $R(\mathbf{n})//H$ with the universal symplectic implosion

$$(T^*K)_{\text{impl}} = (K \times \mathfrak{t}_+)/\sim$$

of $K = SO(n)$, where $(k, \xi) \sim (k', \xi')$ if and only if $\xi = \xi'$, with stabiliser K_ξ under the coadjoint action of K , and $k = k'\tilde{k}$ for some $\tilde{k} \in [K_\xi, K_\xi]$.

We may argue in a very similar way for the symplectic group $Sp(2k, \mathbb{C})$, the complexification of $Sp(2k)$. Following [23] we replace the symmetric bilinear form by the skew form on $\mathbb{C}^n = \mathbb{C}^{2k}$ defined by the matrix

$$J_2 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

Once again we find that $Sp(2k, \mathbb{C})$ acts transitively on the set of α_{r-1} satisfying the condition

$$\alpha_{r-1}^t J_2 \alpha_{r-1} = 0$$

and the above arguments go through mutatis mutandis.

For $SO(n, \mathbb{C})$ with n even, the isotropic subspace $\text{im } \alpha_r$ may be self-dual or anti-self-dual if $n_{r-1} = \frac{n}{2}$. We take the component of the locus defined by (2.2) corresponding to the image being self-dual, and now we get the desired transitivity.

Theorem 2.7. *Let K be a compact classical group and let us consider full flag quivers for $K_{\mathbb{C}}$ as above. That is, we take $\mathbf{n} = (n_1, \dots, n_r)$ to be $(1, 2, \dots, n)$ for $K = SU(n)$, $(1, 2, \dots, k, 2k+1)$ for $SO(2k+1)$, and $(1, 2, \dots, k, 2k)$ for $SO(2k)$ or $Sp(k)$. Also in the orthogonal and symplectic cases we impose the appropriate isotropy condition on the top map in the quiver, and take the appropriate component of the space of isotropic subspaces in the even orthogonal case, to obtain a space $R(\mathbf{n})$ of full flag quivers.*

Then the symplectic quotient of $R(\mathbf{n})$ by $H(\mathbf{n}) = \prod_{i=2}^{r-1} SU(n_i, \mathbb{C})$ can be identified naturally with the universal symplectic implosion for K , or equivalently with the non-reductive GIT quotient $K_{\mathbb{C}}//N$. The

stratification by quiver diagrams as above corresponds to the stratification of the universal symplectic implosion as the disjoint union over the standard parabolic subgroups P of $K_{\mathbb{C}}$ of the varieties $K_{\mathbb{C}}/[P, P]$.

Example 2.8. The lowest rank case of the above construction is when the group is $SO(3)$. The quiver is now just

$$0 \xrightarrow{\alpha_0} \mathbb{C} \xrightarrow{\alpha_1} \mathbb{C}^3$$

where $\alpha_0 = 0$. As $H = SU(1)$ here there is no quotienting to perform. The matrix J is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and, putting $\alpha_1 = (x, y, z)$, the isotropy condition $\alpha_1^t J \alpha_1 = 0$ becomes

$$y^2 + xz = 0.$$

This affine surface is a well known description of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$. This is a valid description of the symplectic implosion for $SO(3)$, since the implosion for the double cover $SU(2)$ is just \mathbb{C}^2 . \diamond

Remark 2.9. We mention here an alternative description of symplectic implosions using the concept of Cox rings [3, 12, 22]. If X is an algebraic variety and L_1, \dots, L_n are generators for $\text{Pic}(X)$, then we form the Cox ring

$$(2.10) \quad \text{Cox}(X, L) = \bigoplus_{(m_1, \dots, m_n) \in \mathbb{Z}^n} H^0(X, m_1 L_1 + \dots + m_n L_n)$$

Hu and Keel [12] introduced the class of *Mori dream spaces* – the varieties X whose Cox ring is finitely generated. These include toric varieties, which are characterised by $\text{Cox}(X)$ being a polynomial ring. It was proved in [12] that, as the name suggests, Mori dream spaces are well behaved from the point of view of the Minimal Model Programme. After a finite sequence of flips and divisorial contractions we arrive at a space birational to X which either is a Mori fibre space or has nef canonical divisor. Mori dream spaces may be realised as GIT quotients by tori of the affine varieties associated to their Cox rings. The above sequence of flips and contractions can be expressed in terms of explicit variation of GIT wall-crossings, and indeed the Mori chambers admit a natural identification with variation of GIT chambers. Since torus variation of GIT is well-understood, in principle the Mori theory of a Mori dream space is also well-understood, at least given an explicit enough presentation of $\text{Cox}(X)$.

If K is a compact Lie group and P is a parabolic subgroup of $K_{\mathbb{C}}$, then the Cox ring of $K_{\mathbb{C}}/P$ is the coordinate ring of the quasi-affine variety $K_{\mathbb{C}}/[P, P]$, which is finitely generated. In particular $K_{\mathbb{C}}/P$ is a Mori dream space.

Taking P to be a Borel subgroup B , we find that the Cox ring of $K_{\mathbb{C}}/B$ is the finitely generated ring $\mathcal{O}(K_{\mathbb{C}}/N) = \mathcal{O}(K_{\mathbb{C}})^N$ whose associated affine variety $\text{Spec}(\mathcal{O}(K_{\mathbb{C}})^N)$ is the universal symplectic implosion $K_{\mathbb{C}}//N$.

3. HYPERKÄHLER QUIVER DIAGRAMS

For $K = SU(n)$ actions we developed in [5] a finite-dimensional approach to constructing the universal hyperkähler implosion for K via quiver diagrams. In that case the symplectic quivers formed a linear space and we just took the cotangent bundle, which amounted to putting in maps $\beta_i: V_{i+1} \rightarrow V_i$ in addition to the $\alpha_i: V_i \rightarrow V_{i+1}$. Writing $V_i = \mathbb{C}^{n_i}$, we thus worked with the flat hyperkähler space

$$(3.1) \quad M = M(\mathbf{n}) = \bigoplus_{i=1}^{r-1} \mathbb{H}^{n_i n_{i+1}} = \bigoplus_{i=1}^{r-1} \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}}) \oplus \text{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i})$$

with the hyperkähler action of $U(n_1) \times \cdots \times U(n_r)$

$$\alpha_i \mapsto g_{i+1} \alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \quad (i = 1, \dots, r-1),$$

with $g_i \in U(n_i)$ for $i = 1, \dots, r$. Right quaternion multiplication was given by

$$(3.2) \quad (\alpha_i, \beta_i) \mathbf{j} = (-\beta_i^*, \alpha_i^*).$$

If each β_i is zero we recovered a symplectic quiver diagram.

We considered the hyperkähler quotient of $M(\mathbf{n})$ with respect to the group $H = \prod_{i=1}^{r-1} SU(n_i)$, obtaining a stratified hyperkähler space $Q = M // H$, which has a residual action of the torus $T^{r-1} = \tilde{H}/H$ where $\tilde{H} = \prod_{i=1}^{r-1} U(n_i)$, as well as a commuting action of $SU(n_r) = SU(n)$. When $\mathbf{n} = (1, 2, \dots, n)$ we can identify this torus with the standard maximal torus T of $SU(n)$ using the simple roots of T .

The *universal hyperkähler implosion for $SU(n)$* is defined to be the hyperkähler quotient $Q = M // H$, where M, H are as above with $n_j = j$, for $j = 1, \dots, n$, (i.e. the case of a full flag quiver).

From the complex-symplectic viewpoint, Q is the GIT quotient, by the complexification

$$H_{\mathbb{C}} = \prod_{i=1}^{r-1} SL(n_i, \mathbb{C})$$

of H , of the zero locus of the complex moment map $\mu_{\mathbb{C}}$ for the H action.

The components of this complex moment map $\mu_{\mathbb{C}}$ are given by the tracefree parts of $\alpha_{i-1} \beta_{i-1} - \beta_i \alpha_i$. The complex moment map equation $\mu_{\mathbb{C}} = 0$ can thus be expressed as saying

$$(3.3) \quad \beta_i \alpha_i - \alpha_{i-1} \beta_{i-1} = \lambda_i^{\mathbb{C}} I \quad (i = 1, \dots, r-1),$$

for some complex scalars $\lambda_1^{\mathbb{C}}, \dots, \lambda_{r-1}^{\mathbb{C}}$, while the real moment map equation is given by

$$(3.4) \quad \beta_{i-1}^* \beta_{i-1} - \alpha_{i-1} \alpha_{i-1}^* - \beta_i \beta_i^* + \alpha_i^* \alpha_i = \lambda_i^{\mathbb{R}} I \quad (i = 1, \dots, r-1),$$

for some real scalars $\lambda_1^{\mathbb{R}}, \dots, \lambda_{r-1}^{\mathbb{R}}$.

The action of $H_{\mathbb{C}}$ is given by

$$\begin{aligned}\alpha_i &\mapsto g_{i+1}\alpha_i g_i^{-1}, & \beta_i &\mapsto g_i\beta_i g_{i+1}^{-1} & (i = 1, \dots, r-2), \\ \alpha_{r-1} &\mapsto \alpha_{r-1} g_{r-1}^{-1}, & \beta_{r-1} &\mapsto g_{r-1}\beta_{r-1},\end{aligned}$$

where $g_i \in SL(n_i, \mathbb{C})$. The residual action of $SL(n, \mathbb{C}) = SL(n_r, \mathbb{C})$ on the quotient Q is given by

$$\alpha_{r-1} \mapsto g_r \alpha_{r-1}, \quad \beta_{r-1} \mapsto \beta_{r-1} g_r^{-1}.$$

There is also a residual action of $\tilde{H}_{\mathbb{C}}/H_{\mathbb{C}}$ which we can identify, in the full flag case, with the maximal torus $T_{\mathbb{C}}$ of $K_{\mathbb{C}}$. The complex numbers λ_i combine to give the complex-symplectic moment map for this complex torus action. We remark that reduction of Q by the maximal torus at level 0 recovers the construction of the nilpotent variety [14], [17]. While there is a similar quiver description of the nilpotent variety for the classical algebras $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$ [14], [18], the construction of an implosion does not directly generalise, partly because the corresponding groups \tilde{H} do not have sufficiently large centers.

Note that as Q is a hyperkähler reduction by H at level 0, it also inherits an $SU(2)$ action that rotates the two-sphere of complex structures (see [5] for details).

Given a quiver $(\alpha, \beta) \in M(\mathbf{n})$, the composition

$$X = \alpha_{r-1}\beta_{r-1} \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n).$$

is invariant under the action of $\tilde{H}_{\mathbb{C}}$ and transforms by conjugation under the residual $SL(n, \mathbb{C})$ action. The map $Q \rightarrow \mathfrak{sl}(n, \mathbb{C})$ given by sending (α, β) to the tracefree part of X is therefore $T_{\mathbb{C}}$ -invariant and $SL(n, \mathbb{C})$ -equivariant.

In [5] and [6] we introduced stratifications of the implosion Q , one reflecting its hyperkähler structure and one reflecting the group structure of $SU(n)$. We recall in particular that the open subset of Q consisting of quivers with all β surjective may be identified with $SL(n, \mathbb{C}) \times_N \mathfrak{n}^0 \cong SL(n, \mathbb{C}) \times_N \mathfrak{b}$.

The open stratum Q^{hks} in the hyperkähler stratification of Q consists of the quivers which are hyperkähler stable; that is, for a generic choice of complex structure all the maps α_i are injective and all the maps β_i are surjective. In this situation the kernels of the compositions

$$\beta_j \circ \beta_{j+1} \circ \dots \circ \beta_{n-1}$$

for $1 \leq j \leq n$ form a full flag in \mathbb{C}^n ; we can use the action of $K = SU(n)$ (which preserves the hyperkähler structure) to put this flag into standard position. Next we can use the action of $SL = H_{\mathbb{C}}$ to put the

maps β_j into the form

$$(3.5) \quad \beta_j = \begin{pmatrix} 0 & \mu_1^j & 0 & \dots & 0 & 0 \\ 0 & 0 & \mu_2^j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_{j-1}^j & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu_j^j \end{pmatrix}$$

for some $\mu_i^j \in \mathbb{C} \setminus \{0\}$. Then it follows from the complex moment map equations (3.3) that the maps α_j have the form

$$(3.6) \quad \alpha_j = \begin{pmatrix} * & * & \dots & * & * \\ \nu_1^j & * & \dots & * & * \\ 0 & \nu_2^j & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \nu_j^{j-1} & * \\ 0 & 0 & \dots & 0 & \nu_j^j \end{pmatrix}$$

and that the same equations (3.3) are satisfied if each α_j is replaced with

$$(3.7) \quad \alpha_j^t = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \nu_1^j & 0 & \dots & 0 & 0 \\ 0 & \nu_2^j & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \nu_j^{j-1} & 0 \\ 0 & 0 & \dots & 0 & \nu_j^j \end{pmatrix}$$

For a fixed choice of complex structures let us denote by $\mathfrak{b}_+^{(\circ)}$ the subset of Q represented by all quivers of the form (3.6) and (3.5) satisfying the hyperkähler moment map equations with μ_i^j and ν_i^j nonzero complex numbers. Its K -sweep $K \mathfrak{b}_+^{(\circ)}$ in Q is then isomorphic to

$$K \times_T \mathfrak{b}_+^{(\circ)} \cong K_{\mathbb{C}} \times_B \mathfrak{b}_+^{(\circ)}$$

and consists of all quivers (2.1) in Q such that for each j the map α_j is injective and the map β_j is surjective and \mathbb{C}^n is the direct sum of

$$\ker(\beta_j \circ \beta_{j+1} \circ \dots \circ \beta_{n-1})$$

and $\text{im}(\alpha_{n-1} \circ \dots \circ \alpha_j)$. It follows that $K \mathfrak{b}_+^{(\circ)}$ is open in Q , and its sweep $SU(2)K \mathfrak{b}_+^{(\circ)}$ under the action of $SU(2)$ which rotates the complex structures (and commutes with the action of K) is the open stratum Q^{hks} of Q .

Associating to a quiver (2.1) in $\mathfrak{b}_+^{(\circ)}$ with the maps α_j and β_j in the form (3.6) and (3.5) the quiver in which α_j is replaced with α_j^T given by (3.7) defines a map ψ from $\mathfrak{b}_+^{(\circ)}$ to the hypertoric variety Q_T defined in [6]. This hypertoric variety is the hyperkähler quotient of the

space M_T of all quivers of the form (3.7) and (3.5) by the action of the maximal torus H_T of H with the induced action of $\tilde{H}_T/H_T = (S^1)^{n-1}$ which is identified with T via the basis of $\mathfrak{t}^* \cong \mathfrak{t}$ corresponding to the simple roots. The image of the map ψ is the open subset $Q_T^{(\circ)}$ of Q_T represented by all quivers of this form with μ_i^j and ν_i^j all nonzero.

The restriction to $\mathfrak{b}_+^{(\circ)}$ of the complex moment map for the action of K associates to a quiver of the form (3.6), (3.5) the upper triangular matrix

$$\alpha_{n-1}\beta_{n-1} - \text{tr}(\alpha_{n-1}\beta_{n-1})\frac{I}{n}$$

and thus takes values in $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$. Combining the map ψ with the projection to \mathfrak{n} of this complex moment map gives us isomorphisms

$$\mathfrak{b}_+^{(\circ)} \cong Q_T^{\circ} \times \mathfrak{n}$$

and

$$K \mathfrak{b}_+^{(\circ)} \cong K \times_T (Q_T^{\circ} \times \mathfrak{n}).$$

Under this identification the complex moment map for T is given by the (T -invariant) complex moment map $\phi: Q_T \rightarrow \mathfrak{t}_{\mathbb{C}}^*$ for the action of T on Q_T , and the complex moment map for K is given by

$$[k, \eta, \zeta] \mapsto \text{Ad}^*k(\phi(\eta) + \zeta)$$

for $k \in K$, $\eta \in Q_T^{\circ}$ and $\zeta \in \mathfrak{n}$.

The hyperkähler moment map for T associates to a quiver satisfying the hyperkähler moment map equations (3.3) and (3.4) the element $(\lambda_1^{\mathbb{C}}, \lambda_1^{\mathbb{R}}, \dots, \lambda_{n-1}^{\mathbb{C}}, \lambda_{n-1}^{\mathbb{R}})$ of $(\mathbb{C} \oplus \mathbb{R})^{n-1}$ identified with $\mathfrak{t}^* \otimes (\mathbb{C} \oplus \mathbb{R}) \cong \mathfrak{t}^* \otimes \mathbb{R}^3$ via the basis of simple roots. The image of its restriction to Q^{hks} is the open subset of $\mathfrak{t}^* \otimes \mathbb{R}^3$ defined by $(\lambda_j^{\mathbb{C}}, \lambda_j^{\mathbb{R}}) \neq (0, 0)$ for $j = 1, \dots, n-1$, while the image of $K \mathfrak{b}_+^{(\circ)}$ is the open subset $(\mathfrak{t}^* \otimes \mathbb{R}^3)^{\circ}$ defined by $\lambda_j^{\mathbb{C}} \neq 0$ for $j = 1, \dots, n-1$. Using the same basis the hyper-toric variety Q_T can be identified with \mathbb{H}^{n-1} and $Q_T^{(\circ)}$ then corresponds to the open subset

$$\{(a_1 + jb_1, \dots, a_{n-1} + jb_{n-1}) \in \mathbb{H}^{n-1} : a_{\ell}, b_{\ell} \in \mathbb{C} \setminus \{0\}\}.$$

Under this identification the hyperkähler moment map $\phi: Q_T^{(\circ)} \rightarrow \mathfrak{t}^* \otimes \mathbb{R}^3$ is given by

$$\begin{aligned} \phi(a_1 + jb_1, \dots, a_{n-1} + jb_{n-1}) = \\ (a_1b_1, |a_1|^2 - |b_1|^2, \dots, a_{n-1}b_{n-1}, |a_{n-1}|^2 - |b_{n-1}|^2); \end{aligned}$$

its fibres are single T -orbits in $Q^{(\circ)}_T$.

From the description of $K \mathfrak{b}_+^{(\circ)}$ above it follows that the hyperkähler moment map for T restricts to a locally trivial fibration

$$Q^{hks} \rightarrow SU(2)(\mathfrak{t}^* \otimes \mathbb{R}^3)^{\circ}$$

over the open subset $SU(2)(\mathfrak{t}^* \otimes \mathbb{R}^3)^{\circ}$ of $\mathfrak{t}^* \otimes \mathbb{R}^3$ with fibre $K \times \mathfrak{n}$.

Similarly the other strata in the hyperkähler stratification of Q are constructed from hyperkähler stable quivers of the form

$$(3.8) \quad 0 \xrightleftharpoons[\beta_0]{\alpha_0} \mathbb{C}^{n_1} \xrightleftharpoons[\beta_1]{\alpha_1} \mathbb{C}^{n_2} \xrightleftharpoons[\beta_2]{\alpha_2} \dots \xrightleftharpoons[\beta_{r-2}]{\alpha_{r-2}} \mathbb{C}^{n_{r-1}} \xrightleftharpoons[\beta_{r-1}]{\alpha_{r-1}} \mathbb{C}^{n_r} = \mathbb{C}^n.$$

Again for generic choices of complex structures for each j the map α_j is injective and the map β_j is surjective and \mathbb{C}^n is the direct sum of

$$\ker(\beta_j \circ \beta_{j+1} \circ \dots \circ \beta_{r-1})$$

and $\text{im}(\alpha_{r-1} \circ \dots \circ \alpha_j)$, and we can use the action of $K = SU(n)$ to put the flag in \mathbb{C}^n defined by the subspaces $\ker(\beta_j \circ \beta_{j+1} \circ \dots \circ \beta_{r-1})$ into standard position. Next we can use the action of $\prod_{j=1}^{r-1} SL(n_j)$ to put the maps β_j into block form of the same shape as (3.5) where now each μ_i^j is a nonzero scalar multiple of an identity matrix. Again it follows from the complex moment map equations (3.3) that the maps α_j have block form similar to (3.6) where each ν_i^j is a nonzero scalar multiple of an identity matrix, and that the same equations (3.3) are satisfied if each α_j is replaced with α_j^T in block diagonal form as at (3.7). We find that the space Q_1^{hks} of hyperkähler quivers of the form (3.8) fibres over an open subset of $\mathfrak{t}_1^* \otimes \mathbb{R}^3$ (where $T_1 \cong (S^1)^{r-1}$) with fibre

$$K \times_{[K \cap P_1, K \cap P_1]} \mathfrak{p}_1^0$$

where P_1 is the standard parabolic in $K_{\mathbb{C}}$ corresponding to the flag defined by the subspaces $\ker(\beta_j \circ \beta_{j+1} \circ \dots \circ \beta_{r-1})$, and \mathfrak{p}_1^0 is the annihilator in $\mathfrak{k}_{\mathbb{C}}^*$ of its Lie algebra \mathfrak{p}_1 . Note that using the standard pairing on $\mathfrak{k}_{\mathbb{C}}$ and identifying \mathfrak{b} with the annihilator \mathfrak{n}^0 of \mathfrak{n} in $\mathfrak{k}_{\mathbb{C}}^*$, we have a projection from $\mathfrak{n} \cong \mathfrak{n}^*$ onto the annihilator of \mathfrak{p}_1 in \mathfrak{n}^* , and this annihilator can be identified with \mathfrak{p}_1^0 since $\mathfrak{n} + \mathfrak{p}_1 = \mathfrak{k}_{\mathbb{C}}$.

By [5, Proposition 6.9] each stratum in Q can be identified with a hyperkähler modification

$$\hat{Q}_1^{hks} = (Q_1^{hks} \times (\mathbb{H} \setminus \{0\})^\ell) // T^\ell$$

of Q_1^{hks} for some Q_1 as above, and the restriction to this stratum \hat{Q}_1^{hks} of the hyperkähler moment map for T is a locally trivial fibration over an open subset of

$$\text{Lie}(Z_K(K \cap P_1))^* \otimes \mathbb{R}^3$$

(where $Z_K(K \cap P_1) \subseteq T$ is the centre of $K \cap P_1$ in K) with fibre

$$K \times_{[K \cap P_1, K \cap P_1]} \mathfrak{p}_1^0.$$

Using the surjection

$$(3.9) \quad K \times \mathfrak{n} \rightarrow K \times_{[K \cap P_1, K \cap P_1]} \mathfrak{p}_1^0$$

induced by the projection $\mathfrak{n} \cong \mathfrak{n}^* \rightarrow \mathfrak{p}_1^0$, we can lift this locally trivial fibration to one with fibre $K \times \mathfrak{n}$ which surjects onto the stratum \hat{Q}_1^{hks} .

In order to patch together these locally trivial fibrations for the different strata, we can blow up the hypertoric variety $Q_T \cong \mathbb{H}^{n-1}$, replacing

it with $\tilde{Q}_T \cong \tilde{\mathbb{H}}^{n-1}$ where $\tilde{\mathbb{H}}^{n-1}$ is the blow-up of $\mathbb{H} \cong \mathbb{C}^2$ at 0 using the complex structure on \mathbb{H} given by right multiplication by i ; this commutes with the hyperkähler complex structures given by left multiplication by i, j and k and also with the action of the S^1 component of the maximal torus $T \cong (S^1)^{n-1}$.

Note that the hyperkähler moment map for the action of S^1 on \mathbb{H} induces an identification of the topological quotient \mathbb{H}/S^1 with \mathbb{R}^3 ; this pulls back to an identification of $\tilde{\mathbb{H}}/S^1$ with the manifold with boundary $\tilde{\mathbb{R}}^3 = (\mathbb{R}^3 \setminus \{0\}) \sqcup S^2$. Let \tilde{Q} be the fibre product

$$\begin{array}{ccc} \tilde{Q} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \mathfrak{t}^* \otimes \tilde{\mathbb{R}}^3 & \longrightarrow & \mathfrak{t}^* \otimes \mathbb{R}^3. \end{array}$$

The descriptions above of the hyperkähler strata of Q as the images of surjections from locally trivial fibrations over subsets of $\mathfrak{t}^* \otimes \mathbb{R}^3$ with fibre $K \times \mathfrak{n}$ patch together to give a locally trivial fibration

$$\hat{Q} \rightarrow \mathfrak{t}^* \otimes \tilde{\mathbb{R}}^3$$

with fibre $K \times \mathfrak{n}$ over the manifold with corners $\tilde{\mathbb{R}}^3$, and surjections $\hat{\chi}: \hat{Q} \rightarrow \tilde{Q}$ and $\chi: \tilde{Q} \rightarrow Q$ where $\hat{\chi}$ collapses fibres via the surjections (3.9) and χ is the pullback of the surjection $\mathfrak{t}^* \otimes \tilde{\mathbb{R}}^3 \rightarrow \mathfrak{t}^* \otimes \mathbb{R}^3$.

Remark 3.10. If at (3.5) we only allow ourselves to use the action of H , not $H_{\mathbb{C}}$, to put the maps β_j into standard form, then we are able to ensure that each β_j is of the form

$$(3.11) \quad \beta_j = \begin{pmatrix} 0 & \mu_1^j & * & \dots & * & * \\ 0 & 0 & \mu_2^j & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_{j-1}^j & * \\ 0 & 0 & 0 & \dots & 0 & \mu_j^j \end{pmatrix}$$

for some $\mu_i^j \in \mathbb{C} \setminus \{0\}$. It still follows from the complex moment map equations (3.3) that the maps α_j then have the form (3.6). Similarly an element of any stratum \hat{Q}_1^{hks} as above can for generic choices of complex structures be put into block form (3.6) and (3.11) using the action of $K \times H$, where now μ_i^j and ν_i^j denote nonzero scalar multiples of identity matrices.

4. PROPERTIES OF HYPERKÄHLER IMPLOSION

In this section we will list some of the main properties of the universal hyperkähler implosion $Q = (T^*K_{\mathbb{C}})_{\text{hkimpl}}$ for $K = SU(n)$ which we expect to be true for more general compact groups K .

i) Q is a stratified hyperkähler space of real dimension $2(\dim K + \dim T)$ where T is a maximal torus of K . It has an action of $K \times T$ which preserves the hyperkähler structure and has a hyperkähler moment map

$$\mu^{K \times T}: Q \rightarrow (\mathfrak{k}^* \oplus \mathfrak{t}^*) \otimes \mathbb{R}^3,$$

as well as a commuting action of $SU(2)$ which rotates the complex structures on Q (see [5] for the case $K = SU(n)$).

ii) The hyperkähler reduction at 0 of Q by T can be identified for any choice of complex structure, via the complex moment map for the action of K , with the nilpotent cone \mathcal{N} in $\mathfrak{k}_{\mathbb{C}}$. We can view this as the statement that the Springer resolution $SL(n, \mathbb{C}) \times_B \mathfrak{n} \rightarrow \mathcal{N}$ is an affinisation map. The reduction at a generic point of $\mathfrak{t}^* \otimes \mathbb{R}^3$ is a semisimple coadjoint orbit of $K_{\mathbb{C}}$, and in general the hyperkähler reduction of Q by T at any point of $\mathfrak{t}^* \otimes \mathbb{R}^3$ can be identified for any choice of complex structure, via the complex moment map for the action of K , with a Kostant variety in $\mathfrak{k}_{\mathbb{C}}^*$ (that is, the closure of a coadjoint orbit). We refer to [5] for the case $K = SU(n)$.

iii) When K is semisimple, simply connected and connected (as for special unitary groups) its universal symplectic implosion embeds in the affine space

$$\bigoplus_{\varpi \in \Pi} V_{\varpi},$$

where $\{V_{\varpi} : \varpi \in \Pi\}$ is the set of fundamental representations of K , as the closure of the $K_{\mathbb{C}}$ -orbit of $v = \sum_{\varpi \in \Pi} v_{\varpi}$ for any choice of highest weight vector v_{ϖ} for the irreducible representation V_{ϖ} . When $K = SU(n)$ it was shown in [7] that the universal hyperkähler implosion Q embeds in the space

$$H^0(\mathbb{P}^1, ((\mathfrak{k}_{\mathbb{C}}^* \oplus \mathfrak{t}_{\mathbb{C}}^*) \otimes \mathcal{O}(2)) \oplus \bigoplus_{\varpi} V_{\varpi} \otimes \mathcal{O}(j(\varpi)))$$

of holomorphic sections of the vector bundle

$$(4.1) \quad \mathcal{V} = ((\mathfrak{k}_{\mathbb{C}}^* \oplus \mathfrak{t}_{\mathbb{C}}^*) \otimes \mathcal{O}(2)) \oplus \bigoplus_{\varpi} V_{\varpi} \otimes \mathcal{O}(j(\varpi))$$

over \mathbb{P}^1 for suitable positive integers $j(\varpi)$. Moreover this embedding induces a holomorphic and generically injective map from the twistor space $\mathcal{Z}Q$ of Q to the vector bundle \mathcal{V} over \mathbb{P}^1 , and the hyperkähler structure can be recovered from this embedding when $K = SU(n)$ ([7]).

iv) Let N be a maximal unipotent subgroup of the complexification $K_{\mathbb{C}}$ of K . It was shown in [5] that when $K = SU(n)$ the algebra of invariants $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ is finitely generated and for any choice of complex structures Q is isomorphic to the affine variety

$$(K_{\mathbb{C}} \times \mathfrak{n}^0) // N = \text{Spec } \mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$$

associated to this algebra of invariants. This variety may be viewed as the complex-symplectic quotient (in the sense of nonreductive GIT) of $T^*K_{\mathbb{C}} = K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$ by the action of N given by $(g, \zeta) \mapsto (gn^{-1}, \text{Ad}(n)\zeta)$.

With respect to this identification the complex moment maps for the commuting K and T actions on Q are the morphisms from $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N$ induced by the N -invariant morphisms from $K_{\mathbb{C}} \times \mathfrak{n}^0$ to $\mathfrak{k}_{\mathbb{C}}^*$ and $\mathfrak{t}_{\mathbb{C}}^*$ given by

$$(g, \zeta) \mapsto \text{Ad}^*(g)\zeta$$

and

$$(g, \zeta) \mapsto \zeta_T$$

where $\zeta_T \in \mathfrak{t}_{\mathbb{C}}^*$ is the restriction of $\zeta \in \mathfrak{n}^0 \subseteq \mathfrak{k}_{\mathbb{C}}^*$ to $\mathfrak{t}_{\mathbb{C}}$.

It has been proved very recently by Ginzburg and Riche [10, Lemma 3.6.2] that the algebra of regular functions on $T^*(G/N)$ is finitely generated for a general reductive G with maximal unipotent subgroup N . Taking $G = K_{\mathbb{C}}$ for any compact group K this cotangent bundle may be identified with $K_{\mathbb{C}} \times_N \mathfrak{n}^0$, and its algebra of regular functions is $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$. Hence the non-reductive GIT quotient $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N = \text{Spec } \mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ is a well defined affine variety in general, and is the canonical affine completion of the quasi-affine variety $T^*(K_{\mathbb{C}}/N)$ just as $K_{\mathbb{C}} // N$ is the canonical affine completion of the quasi-affine variety $K_{\mathbb{C}}/N$. It is enough to consider the case when K is semisimple, connected and simply connected. Then their proof provides a reasonably explicit set of generators involving the fundamental representations V_{ϖ} of K and these give an embedding of the affine variety $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N = \text{Spec } \mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ as a closed subvariety of the space of sections $H^0(\mathbb{P}^1, \mathcal{V})$ of a vector bundle \mathcal{V} over \mathbb{P}^1 as at (4.1) above. Note also that the GIT complex-symplectic quotient at level 0 of $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N$ may be viewed as $(K_{\mathbb{C}} \times \mathfrak{n}) // B$ which is the nilpotent variety (see the remarks in ii) above). Similarly reductions at other levels will yield the Kostant varieties (cf. the discussion in §3 of [5]).

Thus we expect that in general, as in the case when $K = SU(n)$, $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N$ has a hyperkähler structure determined by this embedding and can be identified with the universal hyperkähler implosion for K .

Note that the scaling action of \mathbb{C}^* on \mathfrak{n}^0 induces an action of \mathbb{C}^* on $K_{\mathbb{C}} \times \mathfrak{n}^0$ which commutes with the action of N and thus induces an action of \mathbb{C}^* on $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N$. Since \mathbb{C}^* acts on $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)$, and thus on $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$, with only non-negative weights, the sum of the strictly positive weight spaces forms an ideal I in $\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N$ which defines the fixed point set for the action of \mathbb{C}^* on $(K_{\mathbb{C}} \times \mathfrak{n}^0) // N$. This fixed point set is therefore the affine variety $\text{Spec}(\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N / I)$, which can be naturally identified with $\text{Spec}((\mathcal{O}(K_{\mathbb{C}} \times \mathfrak{n}^0)^N)^{\mathbb{C}^*})$ and thus with the universal symplectic implosion $K_{\mathbb{C}} // N = \text{Spec}(\mathcal{O}(K_{\mathbb{C}})^N)$.

v) If $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{k}^* \otimes \mathbb{R}^3$ let

$$K_{\zeta} = K_{\zeta_1} \cap K_{\zeta_2} \cap K_{\zeta_3}$$

where K_{ζ_j} is the stabiliser of ζ_j under the coadjoint action, and let \mathcal{N}_{ζ} be the nilpotent cone in $(\mathfrak{k}_{\zeta})_{\mathbb{C}}^*$ which we identify with $(\mathfrak{k}_{\zeta})_{\mathbb{C}}$ as usual.

By work of Kronheimer [20] there is a $K_\zeta \times T \times SU(2)$ -equivariant embedding

$$\mathcal{N}_\zeta \rightarrow \mathfrak{k}_\zeta \otimes \mathbb{R}^3$$

whose composition with the projection from $\mathfrak{k}_\zeta \otimes \mathbb{R}^3$ to $(\mathfrak{k}_\zeta)_\mathbb{C}$ for any choice of complex structures is the inclusion of the nilpotent cone \mathcal{N}_ζ in $(\mathfrak{k}_\zeta)_\mathbb{C}$. From the discussion in §3 we expect that for any compact group K the image of the hyperkähler moment map for the action of K on the universal hyperkähler implosion Q should be the K -sweep of

$$\mathfrak{t}_{(\text{hk})} = \{ \zeta + \xi \in \mathfrak{k} \otimes \mathbb{R}^3 : \zeta \in \mathfrak{t} \otimes \mathbb{R}^3 \text{ and } \xi \in \mathcal{N}_\zeta \}$$

and the hyperkähler implosion $X_{\text{hkimpl}} = (X \times Q) // K$ for any hyperkähler manifold X with a Hamiltonian hyperkähler action of K and hyperkähler moment map $\mu_X : X \rightarrow \mathfrak{k} \otimes \mathbb{R}^3$ should be given by

$$X_{\text{hkimpl}} = \mu_X^{-1}(\mathfrak{t}_{(\text{hk})}) / \sim .$$

Here $x \sim y$ if and only if $\mu_X(x) = \zeta + \xi$ and $\mu_X(y) = w(\zeta + \xi')$ for some $\zeta \in \mathfrak{t} \otimes \mathbb{R}^3$, some $\xi, \xi' \in \mathcal{N}_\zeta \subseteq \mathfrak{k} \otimes \mathbb{R}^3$, some w in the Weyl group W of K , identified with a finite subgroup of the normaliser of T in K , and moreover $x = kw^{-1}y$ for some $k \in [K_\zeta, K_\zeta]$.

5. HYPERKÄHLER IMPLOSION FOR SPECIAL ORTHOGONAL AND SYMPLECTIC GROUPS

In the case of $K = SU(n)$ the quiver model for the universal symplectic implosion is a symplectic quotient of a flat linear space, so to obtain a quiver model for the universal hyperkähler implosion we could take its cotangent bundle (replacing symplectic with hyperkähler quivers) and the corresponding hyperkähler quotient.

We would like to mimic this construction for the orthogonal and symplectic groups. However we now have the problem that the space of symplectic quivers has a non-flat piece since the top map α_{r-1} has to satisfy the system of quadrics (2.2) given by $\alpha_{r-1}^t J \alpha_{r-1} = 0$.

If this system of equations cut out a smooth variety we could appeal to a result of Feix [9] (see also Kaledin [13]) that gives a hyperkähler structure on an open neighbourhood of the zero section of the cotangent bundle of a Kähler manifold with real-analytic metric. In our case, however, the variety defined by (2.2) is singular. We could of course stratify into smooth varieties by the rank of α_{r-1} and apply Feix's result stratum by stratum, but to obtain a suitable hyperkähler thickening a more global approach is required.

The discussion in §3 and §4 suggests that we should consider first what the analogue of the hypertoric variety Q_T might be when K is a symplectic or special orthogonal group. As in §2 let us first consider the case of $K = SO(n)$ when $n = 2r - 1$ is odd.

For the universal symplectic implosion in this case we considered symplectic quivers

$$(5.1) \quad 0 \xrightarrow{\alpha_0} \mathbb{C} \xrightarrow{\alpha_1} \mathbb{C}^2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{r-2}} \mathbb{C}^{r-1} \xrightarrow{\alpha_{r-1}} \mathbb{C}^n$$

and imposed the constraint $\alpha_{r-1}^t J \alpha_{r-1} = 0$; we then took the symplectic quotient by $H_r = \prod_{j=1}^{r-1} SU(j)$ with respect to the standard Kähler structure on $(\mathbb{C}^{j-1})^* \otimes \mathbb{C}^j$ for $j \leq r-1$ and the Kähler structure induced by J on $(\mathbb{C}^{r-1})^* \otimes \mathbb{C}^n \cong (\mathbb{C}^n)^{r-1}$. We saw that there is a natural map to this symplectic quotient from the toric variety given by the symplectic quotient of the space of symplectic quivers as above where each map α_j has the form

$$\alpha_j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \nu_1^j & 0 & \dots & 0 \\ 0 & \nu_2^j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_j^j \end{pmatrix}$$

for $j < r-1$ and

$$\alpha_{r-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \nu_1^{r-1} & 0 & \dots & 0 \\ 0 & \nu_2^{r-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_{r-1}^{r-1} \end{pmatrix}$$

with $\nu_i^j \in \mathbb{C}$ as at (2.5); notice that a quiver of this form always satisfies the constraint $\alpha_{r-1}^t J \alpha_{r-1} = 0$.

By analogy with this and with the hypertoric variety Q_T described in §3 for the case when $K = SU(n)$, we expect the hypertoric variety Q_T for $K = SO(n)$ when $n = 2r-1$ to be closely related to the hyperkähler quotient by the standard maximal torus T_{H_r} of H_r of the flat space $M_T^{SO(n)}$ given by quiver diagrams

$$(5.2) \quad 0 \xrightleftharpoons[\beta_0]{\alpha_0} \mathbb{C} \xrightleftharpoons[\beta_1]{\alpha_1} \mathbb{C}^2 \xrightleftharpoons[\beta_2]{\alpha_2} \dots \xrightleftharpoons[\beta_{r-2}]{\alpha_{r-2}} \mathbb{C}^{r-1} \xrightleftharpoons[\beta_{r-1}]{\alpha_{r-1}} \mathbb{C}^n$$

where the maps α_j and β_j have the form

$$\alpha_j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \nu_1^j & 0 & \dots & 0 \\ 0 & \nu_2^j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_j^j \end{pmatrix} \quad \text{and} \quad \beta_j = \begin{pmatrix} 0 & \mu_1^j & 0 & \dots & 0 & 0 \\ 0 & 0 & \mu_2^j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_{j-1}^j & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu_j^j \end{pmatrix}$$

if $j < r - 1$ and

$$\alpha_{r-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ \nu_1^{r-1} & 0 & \dots & 0 \\ 0 & \nu_2^{r-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_{r-1}^{r-1} \end{pmatrix}$$

and

$$\beta_{r-1} = \begin{pmatrix} 0 & \dots & 0 & \mu_1^{r-1} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & \mu_2^{r-1} & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \mu_{r-2}^{r-1} & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mu_{r-1}^{r-1} \end{pmatrix}$$

for some $\nu_i^j, \mu_i^j \in \mathbb{C} \setminus \{0\}$. Notice that

$$(5.3) \quad \alpha_{r-1}^t J \alpha_{r-1} = 0 = \beta_{r-1} J \beta_{r-1}^t$$

for any α_{r-1} and β_{r-1} of this form.

Let $M^{SO(n)}$ be the flat hyperkähler space given by arbitrary quiver diagrams of the form (5.2), where the hyperkähler structure is induced by the standard hyperkähler structure on $(\mathbb{C}^{j-1})^* \otimes \mathbb{C}^j \oplus (\mathbb{C}^j)^* \otimes \mathbb{C}^{j-1} \cong \mathbb{H}^{(j-1)j}$ for $j \leq r - 1$ and the hyperkähler structure induced by J on $(\mathbb{C}^{r-1})^* \otimes \mathbb{C}^n \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^{r-1} \cong (\mathbb{H}^n)^{r-1}$. As in the symplectic case discussed in §2, the restriction to $M_T^{SO(n)}$ of the hyperkähler moment map for the action of H_r coincides with the hyperkähler moment map for the action of T_{H_r} on $M_T^{SO(n)}$. Thus

$$Q_T^{SO(n)} = M_T^{SO(n)} // T_{H_r}$$

maps naturally to $M^{SO(n)} // H_r$.

By analogy with the discussion in §4 we can consider the subset of $M^{SO(n)} // H_r$ which is the closure of the $K_{\mathbb{C}} = SO(n, \mathbb{C})$ -sweep of the image of $Q_T^{SO(n)}$. By (5.3) this is contained in the closed subset defined by the $K_{\mathbb{C}}$ -invariant constraints

$$\alpha_{r-1}^t J \alpha_{r-1} = 0 = \beta_{r-1} J \beta_{r-1}^t,$$

and this closed subset has the dimension expected of the universal hyperkähler implosion. Thus we expect the subset of the hyperkähler quotient $M^{SO(n)} // H_r$ defined by these constraints to be closely related to the universal hyperkähler implosion for $K = SO(n)$ when $n = 2r - 1$ is odd. Similarly we expect that modifications of this construction as described in §2 for the universal symplectic implosion will be closely related to the universal hyperkähler implosion for the special orthogonal groups $K = SO(n)$ when n is even and for the symplectic groups. The

following example, however, provides a warning against over-optimism here.

Example 5.4. Recall that $SO(3) = SU(2)/\{\pm 1\}$ and that the universal symplectic implosion for $SO(3)$ is $\mathbb{C}^2/\{\pm 1\}$, where \mathbb{C}^2 is the universal symplectic implosion for $SU(2)$. Moreover the universal hyperkähler implosion for $SU(2)$ is \mathbb{H}^2 , given by quiver diagrams of the form

$$(5.5) \quad \mathbb{C} \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \mathbb{C}^2,$$

(recall that the group H_1 here is trivial, as in Example 2.8, so no quotienting occurs). We thus expect the universal hyperkähler implosion for $SO(3)$ to be $\mathbb{H}^2/\{\pm 1\}$.

We can associate to any quiver (5.5) the quiver

$$\mathbb{C} \cong \mathrm{Sym}^2(\mathbb{C}) \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{C}^3$$

where $\alpha_1 = \mathrm{Sym}^2(\alpha)$ and $\beta_1 = \mathrm{Sym}^2(\beta)$ are the maps between $\mathrm{Sym}^2(\mathbb{C})$ and $\mathrm{Sym}^2(\mathbb{C}^2)$ induced by α and β . This construction gives us a surjection from \mathbb{H}^2 to the subvariety of the space $M^{SO(3)} // H_1 = M^{SO(3)}$ of quivers

$$\mathbb{C} \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} \mathbb{C}^3$$

satisfying $\alpha_1^t J \alpha_1 = 0 = \beta_1 J \beta_1^t$, but it gives an identification of this subvariety with the quotient of \mathbb{H}^2 by $\mathbb{Z}_2 \times \mathbb{Z}_2$, not by $\mathbb{Z}_2 = \{\pm 1\}$. \diamond

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